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2007 J. Phys. A: Math. Theor. 40 979

(http://iopscience.iop.org/1751-8121/40/5/008)

Institute of Physics Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **40** (2007) 979–993 doi:10.1088/1751-8113/40/5/008

Curve motions from the integrable equations having variable spectral parameters

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Received 2 October 2006, in final form 23 November 2006
Published 17 January 2007
Online at stacks.iop.org/JPhysA/40/979

Abstract
This paper reports on a study of the motion of moving space curves induced from integrable equations having variable spectral parameters. We explain the geometric structure of the curves using the associated linear equations of integrable equations. A generalized form of the Hasimoto transformation is introduced. This form relates the curvature and torsion of the curves to the variables of integrable equations. Some explicit curve motions, including the solitonic and Kelvin-type curves, are calculated using the Sym–Tafel formula.

PACS numbers: 47.32.Cc, 02.30.Gp, 67.57.Fg

1. Introduction

There are many nonlinear problems in physics that can be described by moving helical space curves [1]. Interestingly, some important curve motions are related to the famous integrable equations. For example, an isolated thin vortex filament in a fluid can be described by the nonlinear Schrödinger equation (NLSE) through the Hasimoto transformation [2]. The Hasimoto transformation relates the curvature and torsion of curves to the amplitude and phase of the variable of integrable equations. Another example is the problem of spin dynamics of the Heisenberg ferromagnet(HF) model. Lakshmanan was first to show that the continuum dynamics of the HF model is gauge equivalent to the NLSE [3]. Here, spin motion is related to the tangent vector of the curve of the vortex filament. Lamb extended the result obtained by Hasimoto. Lamb investigated the connection between the sine-Gordon and Hirota equations with the motion of certain helical curves [4]. More recently, [5] and [6] related the modified Korteweg–de Vries (mKdV) equation to motions of curves in a plane. In all of these studies, the Hasimoto transformation was important in providing connections between the nonlinear dynamics of moving curves and the corresponding integrable equations.

Another interesting development in investigating the problems of moving space curves is the Sym–Tafel formula. This formula associates the coordinate of space curves with the solution of the linear problem of corresponding integrable equations [7]. Various problems of space curves mentioned above have been reformulated using the Sym–Tafel formula. Also, some explicit form of curves has been constructed using this formula [8, 9]. It includes all the previously mentioned problems as well as the Lund–Regge vortex motions which have been described by the complex sine-Gordon equation [10, 11].

The construction of the completely integrable inhomogeneous equations has been an important development in nonlinear problems [12, 13]. Obviously, they are more realistic and appropriate in describing real physical situations [14–16] such as the problems of space curves moving in an inhomogeneous and/or imperfectly smooth space. Some inhomogeneous integrable equations have been constructed using various methods such as (1) integrable equations having a variable spectral parameter [17, 18] or (2) guessing integrable equations having inhomogeneous terms and then checking the integrability of them using the Painlevè test [18–20], or (3) Darboux covariant construction of inhomogeneous integrable equations [21, 22]. These inhomogeneous theories were applied to real nonlinear problems, for example, the dc and ac conductivity of one-dimensional condensates described by the so-called damped, driven NLSE, and the reduced Maxwell–Bloch system with pumping or with a damping of a special type.

It would be interesting to consider space curve problems, which are related to inhomogeneous integrable equations. In fact, publications have already appeared on some studies of this topic. A classical inhomogeneous Heisenberg chain having site-dependent interaction has been shown to be described by an integro-differential NLSE [23]. More recently, using the Lamb formalism, Porsezian systematically derived various NLSE-type integrable equations such as derivative NLSE, higher order NLSE, inhomogeneous NLSE, and inhomogeneous radially symmetric NLSE, which can be connected with certain inhomogeneous curve problems [24]. However, their corresponding curve equations, as well as their associated linear equations, were not given explicitly in [24]. In this respect, a more systematic investigation of the inhomogeneous curve problems would be required.

In our research and now presented in this paper, we have used the Sym–Tafel formula to investigate curve motions moving in an inhomogeneous space, starting from the associated linear problems of integrable equations having variable spectral parameter. Equations of curve motion, which have specific spacetime dependence, have been constructed. Some of their solutions have been calculated, including the Kelvin-type and solitonic curves.We introduce a generalized form of Hasimoto transformation. This shows that the constructed curves have nontrivial curvature and torsion. Torsions have spacetime dependences, which are inherited from the spacetime dependence of variable spectral parameters. By using the Lamb formalism, we show that the space curve problems can induce various integrable equations having variable spectral parameter. We note that there is a soliton surface theory constructed on a integrable equation having variable spectral parameter [25]. Though the spirit of [25] is similar to ours, the results of our present research presented in this paper, including a generalized Hasimoto transformation, contain unique features from the approach of the space curve problem.

2. Space curves from integrable equations having variable spectral parameter

2.1. Integrable equations having variable spectral parameter

Consider a two-dimensional integrable system defined by the Lax pair in terms of 2×2 matrices $U_i(\psi)$, $V_i(\psi)$:

$$\begin{aligned} \partial_x \Phi &= \sum_{m=0}^{1} \lambda(x,t)^m U_m \Phi \equiv U\Phi, \\ \partial_t \Phi &= \sum_{n=-a}^{b} \lambda(x,t)^n V_n \Phi \equiv V\Phi, \end{aligned} \tag{1}$$

where $\partial_x \equiv \partial/\partial x$, $\partial_t \equiv \partial/\partial t$ and $\lambda(x,t)$ is the variable spectral parameter. In this paper, we consider the case where

$$U_1 = \mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad U_0 = \begin{pmatrix} 0 & \Psi \\ -\Psi^* & 0 \end{pmatrix}, \tag{2}$$

such that

$$U = \mathrm{i}\lambda(x,t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & \Psi \\ -\Psi^* & 0 \end{pmatrix}. \tag{3}$$

Here $\Psi(x,t)$ denotes the variable of integrable equations. Integers a, b in equation (1) are chosen appropriately for specific integrable equation. The variable spectral parameter $\lambda(x,t)$ satisfies certain relations of the type

$$\partial_x \lambda(x,t) = \sum_{m=0}^{1} \lambda(x,t)^m \alpha_m, \qquad \partial_t \lambda(x,t) = \sum_{n=-a}^{b} \lambda(x,t)^n \beta_n \tag{4}$$

and thus has explicit dependence on x and t. The compatibility of two equations in equation (1) for any value of $\lambda(x,t)$ requires

$$\partial_t U_k - \partial_x V_k + \sum_{m=-\infty}^{\infty} ([U_m, V_{k-m}] + m\beta_{k+1-m} U_m - m\alpha_{k+1-m} V_m) = 0. \tag{5}$$

Here we have extended our notation so that $U_m = \alpha_m = 0$, for $m < 0$ or $m > 1$, and $V_n = \beta_n = 0$ for $n < -a$ or $n > b$. Equation (5) gives constraints on V_m, α_m, β_m for $k \neq 0$ (or $k \neq -1$ for Maxwell–Bloch equation), while it becomes the equation of motion at $k = 0$ (or $k = -1$ for Maxwell–Bloch equation).

Here we give some examples of V_m, α_m, β_m which will be considered in this paper [12].

Example 1: the NLSE

For this equation, we take $a = 0$, $b = 2$ and

$$V = -2\lambda(x,t)^2 U_1 - 2\lambda(x,t) U_0 + \begin{pmatrix} \mathrm{i}|\Psi|^2 + 2\mathrm{i}\int \frac{|\Psi|^2}{x}\,\mathrm{d}x & \mathrm{i}\partial_x \Psi + \mathrm{i}\frac{\Psi}{x} \\ \mathrm{i}\partial_x \Psi^* + \mathrm{i}\frac{\Psi^*}{x} & -\mathrm{i}|\Psi|^2 - 2\mathrm{i}\int \frac{|\Psi|^2}{x}\,\mathrm{d}x \end{pmatrix}. \tag{6}$$

The variable spectral parameter satisfies equation (4) with $\alpha_1 = 1/x$, $\beta_2 = -4/x$, $\alpha_0 = \beta_1 = \beta_0 = 0$, i.e.,

$$\partial_x \lambda(x,t) = \frac{\lambda(x,t)}{x}, \qquad \partial_t \lambda(x,t) = -\frac{4}{x}\lambda(x,t)^2. \tag{7}$$

Thus $\lambda(x,t)$ is given by

$$\lambda(x,t) = \frac{x}{4(\mu + t)}, \tag{8}$$

where μ is a constant, which is the hidden spectral parameter. In the present consideration, $\lambda(x,t)$ is related to the torsion of curves, and it should take a real value. In the same reason,

μ takes also a real value. The compatibility equation (5) is satisfied at $k = 1, 2$ with V_m, α_m, β_m in equations (4), (7), while it becomes the NLSE having variable spectral parameter at $k = 0$;

$$\mathrm{i}\partial_t\Psi + \partial_x^2\Psi + \frac{1}{x}\partial_x\Psi + 2|\Psi|^2\Psi - \frac{\Psi}{x^2} + 4\Psi\int\frac{|\Psi|^2}{x}\,\mathrm{d}x = 0. \tag{9}$$

Example 2: the mKdV equation

For this equation, we take $a = 0, b = 3$, and Ψ is taken to be real, $\Psi = \Psi^*$. The V matrix is

$$\begin{aligned} V = 4\lambda(x,t)^3 xU_1 + 4\lambda(x,t)^2 xU_0 + 2\mathrm{i}\lambda(x,t)\begin{pmatrix} -\partial_x\left(x\int\Psi^2\,\mathrm{d}x\right) & -\partial_x(x\Psi) \\ -\partial_x(x\Psi) & \partial_x\left(x\int\Psi^2\,\mathrm{d}x\right)\end{pmatrix} \\ + \begin{pmatrix} 0 & -\partial_x^2(x\Psi) - 2\Psi\partial_x\left(x\int\Psi^2\,\mathrm{d}x\right) \\ \partial_x^2(x\Psi) + 2\Psi\partial_x\left(x\int\Psi^2\,\mathrm{d}x\right) & 0\end{pmatrix}. \end{aligned} \tag{10}$$

Equation (4) in this case is given by ($\alpha_i = 0, \beta_3 = 4, \beta_2 = \beta_1 = \beta_0 = 0$)

$$\partial_x\lambda(x,t) = 0, \qquad \partial_t\lambda(x,t) = 4\lambda(x,t)^3, \tag{11}$$

such that

$$\lambda(x,t) = \frac{1}{\sqrt{-8(\mu+t)}}. \tag{12}$$

In the present formalism, $\lambda(x,t)$ is real and $t < -\mu$. The compatibility equation (5) at $k = 0$ becomes the mKdV equation with variable spectral parameter;

$$\partial_t\Psi + \partial_x^3(x\Psi) + 2\partial_x\left(\Psi\partial_x\left(x\int\Psi^2\,\mathrm{d}x\right)\right) = 0, \tag{13}$$

while other compatibility equations are satisfied with V in equation (10) and $\lambda(x,t)$ in equation (11).

Example 3: the Maxwell–Bloch equation

This equation corresponds to taking $a = 1, b = -1$ and

$$V = \frac{1}{4\mathrm{i}\lambda(x,t)}\begin{pmatrix}\Delta & \rho \\ \rho^* & -\Delta\end{pmatrix} \equiv \frac{1}{4\mathrm{i}\lambda(x,t)}M, \tag{14}$$

where Δ and ρ denote the probability amplitude of the ground state and excited state respectively. Ψ denotes the optical pulse. Reference [12, 26] introduces various possible forms for α_m, β_m. Here, as an example, we take

$$\partial_x\lambda(x,t) = 0, \qquad \partial_t\lambda = \frac{c}{\lambda(x,t)}, \tag{15}$$

such that

$$\lambda(x,t) = \sqrt{\mu + 2ct}. \tag{16}$$

c is a constant related to the pumping in the Maxwell–Bloch system. The compatibility equation (5) at $k = 0$ and $k = -1$ becomes the Maxwell–Bloch equation with variable spectral parameter;

$$2\partial_t\Psi + \rho = 0, \qquad \partial_x\Delta + 4c - (\rho\Psi^* + \rho^*\Psi) = 0, \qquad \partial_x\rho + 2\Delta\Psi = 0. \tag{17}$$

2.2. The Serret–Frenet equation from the Sym–Tafel formula

Now we introduce a generalized form of Sym–Tafel formula, which writes down the curve variable $\mathbf{r}$ in terms of Φ in equation (1) [7, 8]. Then we show that one of the Lax equation, $\partial_x \Phi = U\Phi$, implies the Serret–Frenet equation. Originally, the Sym–Tafel formula was introduced to integrate the moving frame of integrable surfaces of given geometric properties. In the present formalism, it will be used to relate Φ with a point on a single curve in space at a time. First we introduce two functions $J(x)$ and $K(t)$, which results from

$$\frac{\partial\lambda(x,t)}{\partial\mu} = J(x)K(t), \tag{18}$$

where μ is the hidden spectral parameter in equations (8), (12) and (16). Let us define a matrix r from the position vector $\mathbf{r} \equiv (r_1, r_2, r_3)$ such that $r \equiv \sum r_i\sigma_i$, $i = 1, 3$, where σ_i are Pauli matrices. (From now on, we will omit the summation notation. Please distinguish between the vectors $\mathbf{r}$, $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ and the matrices r, $\hat{t}$, $\hat{n}$, $\hat{b}$ in the following.) Let us relate r with Φ as

$$r = -\mathrm{i}\frac{1}{K(t)}\Phi^{-1}\frac{\partial}{\partial\mu}\Phi = -\mathrm{i}J(x)\Phi^{-1}\frac{\partial}{\partial\lambda}\Phi. \tag{19}$$

To check that it gives the Serret–Frenet equation, we first introduce a new variable $s = \int J(x)\,\mathrm{d}x$. Now note that

$$\begin{aligned}\partial_s r \equiv \frac{\partial}{\partial s}r = \frac{1}{J(x)}\partial_x r &= \frac{-\mathrm{i}}{J(x)K(t)}\left(-\Phi^{-1}\partial_x\Phi\Phi^{-1}\frac{\partial}{\partial\mu}\Phi + \Phi^{-1}\frac{\partial}{\partial\mu}(\partial_x\Phi)\right)\\ &= \frac{-\mathrm{i}}{J(x)K(t)}\Phi^{-1}\frac{\partial U}{\partial\mu}\Phi = \Phi^{-1}\sigma_3\Phi,\end{aligned} \tag{20}$$

where we use the Lax equation $\partial_x\Phi = U\Phi$ in the last step. In other words, the unit tangent vector $\mathbf{t} = \partial_s\mathbf{r} \equiv (t_1, t_2, t_3)$ is given by

$$\hat{t} \equiv t_i\sigma_i = \partial_s r_i\sigma_i = \partial_s r = \Phi^{-1}\sigma_3\Phi. \tag{21}$$

Note that $t_i t_i = \frac{1}{2}\,\mathrm{Tr}\,\hat{t}^2 = 1$. This equation shows that the tangent vector $\mathbf{t}$ is given by the rotation of $\hat{k}$ (the unit vector along the z-axis), where the rotation is induced by the similarity transformation of Φ. In a similar way, we find that

$$\partial_s\hat{t} = \partial_s^2 r = -[\Phi^{-1}\partial_s\Phi, \Phi^{-1}\sigma_3\Phi] = \frac{1}{J(x)}\Phi^{-1}[U, \sigma_3]\Phi. \tag{22}$$

Now, we parametrize the variable Ψ as

$$\Psi = \tfrac{1}{2}J(x)\kappa(x,t)\exp[\mathrm{i}\theta(x,t)]. \tag{23}$$

Then equation (22) becomes

$$\partial_s\hat{t} = \kappa(x,t)\Phi^{-1}\begin{pmatrix} 0 & \exp[\mathrm{i}\theta(x,t)]\\ \exp[-\mathrm{i}\theta(x,t)] & 0\end{pmatrix}\Phi. \tag{24}$$

Equation (24) is one of the Serret–Frenet equations, $\partial_s\mathbf{t} = \kappa\mathbf{n}$ (κ is the curvature), when we define the normal vector $\mathbf{n} \equiv (n_1, n_2, n_3)$ as

$$\hat{n} \equiv n_i\sigma_i = \Phi^{-1}\begin{pmatrix} 0 & \exp[\mathrm{i}\theta(x,t)]\\ \exp[-\mathrm{i}\theta(x,t)] & 0\end{pmatrix}\Phi. \tag{25}$$

Thus, the normal vector $\mathbf{n}$ is given by the rotation of the following unit vector:

$$\cos\theta\hat{i} - \sin\theta\hat{j}, \tag{26}$$

($\hat{i}$, $\hat{j}$ are the unit vectors along the x and y-axes, respectively) induced by the similarity transformation of Φ. It is clear that $\mathbf{t}$ and $\mathbf{n}$ are orthogonal to each other. Now,

$$\begin{aligned}\partial_s\hat{n} &= \frac{1}{J(x)}\Phi^{-1}\left(\left[\begin{pmatrix}0 & \exp[\mathrm{i}\theta(x,t)]\\ \exp[-\mathrm{i}\theta(x,t)] & 0\end{pmatrix}, U\right]\right.\\ &\quad\left. + \mathrm{i}\partial_x\theta\begin{pmatrix}0 & \exp[\mathrm{i}\theta(x,t)]\\ \exp[-\mathrm{i}\theta(x,t)] & 0\end{pmatrix}\right)\Phi\\ &= \frac{2\lambda(x,t)-\partial_x\theta}{J(x)}\Phi^{-1}\begin{pmatrix}0 & -\mathrm{i}\exp[\mathrm{i}\theta(x,t)]\\ \mathrm{i}\exp[-\mathrm{i}\theta(x,t)] & 0\end{pmatrix} - \kappa\Phi^{-1}\sigma_3\Phi,\end{aligned} \tag{27}$$

which gives another Serret–Frenet equation $\partial_s\mathbf{n} = \tau\mathbf{b} - \kappa\mathbf{t}$, when we take the binormal vector $\mathbf{b} \equiv (b_1, b_2, b_3)$ as

$$\hat{b} \equiv b_i\sigma_i = \Phi^{-1}\begin{pmatrix}0 & \mathrm{i}\exp[\mathrm{i}\theta(x,t)]\\ -\mathrm{i}\exp[-\mathrm{i}\theta(x,t)] & 0\end{pmatrix}\Phi, \tag{28}$$

and the torsion

$$\tau = \frac{\partial_x\theta - 2\lambda(x,t)}{J(x)}. \tag{29}$$

In this case, the binormal vector $\mathbf{b}$ is given by the rotation of the following unit vector:

$$\mathbf{b} = -\sin\theta\hat{i} - \cos\theta\hat{j}. \tag{30}$$

A distinguishing feature of the present formalism compared to the previous studies [23, 24] is the appearance of spectral parameter $\lambda(x,t)$ in the torsion τ. Finally, similar calculation gives

$$\partial_s\hat{b} = \frac{2\lambda(x,t)-\partial_x\theta}{J(x)}\Phi^{-1}\begin{pmatrix}0 & \exp[\mathrm{i}\theta(x,t)]\\ \exp[-\mathrm{i}\theta(x,t)] & 0\end{pmatrix}\Phi, \tag{31}$$

which gives another Serret–Frenet equation, $\partial_s\mathbf{b} = -\tau\mathbf{n}$. Note that all these relations show the variable $s = \int J(x)\,\mathrm{d}x$ that correponds to the arc length along the curve constructed by the Sym–Tafel formula in equation (19).

2.3. *Equation of motion of space curves*

The time dependence of r can be calculated as

$$\begin{aligned}\partial_t r &= -\mathrm{i}\partial_t\left(\frac{1}{K(t)}\Phi^{-1}\frac{\partial}{\partial\mu}\Phi\right) = -[\partial_t\ln K(t)]r - \mathrm{i}\frac{1}{K(t)}\Phi^{-1}\frac{\partial V}{\partial\mu}\Phi\\ &= -[\partial_t\ln K(t)]r - \mathrm{i}J(x)\Phi^{-1}\frac{\partial V}{\partial\lambda}\Phi.\end{aligned} \tag{32}$$

Here we give the equations of motion for curves associated with the integrable equations considered in the previous sections.

Example 1: the NLSE

By using equations (18) and (8), we can find $J(x) = x$, $K(t) = -1/[4(\mu+t)^2]$ and $\Psi = \frac{1}{2}x\kappa(x,t)\exp(\mathrm{i}\theta)$. Then equations (32) and (6) gives

$$\partial_t r = \frac{2}{\mu+t}r - 4\lambda(x,t)x\hat{t} + x^2\kappa(x,t)\hat{b}, \tag{33}$$

where $\hat{t}$ and $\hat{b}$ are given by equations (21) and (28). This equation can be written in terms of the curve variable $\mathbf{r}$,

$$\partial_t\mathbf{r} = \frac{2}{\mu+t}\mathbf{r} - 4\lambda x\partial_s\mathbf{r} + x^2\partial_s^2\mathbf{r}\times\partial_s\mathbf{r} = \frac{2}{\mu+t}\mathbf{r} - \frac{x}{\mu+t}\partial_x\mathbf{r} + \frac{1}{x}\partial_x^2\mathbf{r}\times\partial_x\mathbf{r}. \tag{34}$$

This type of equation has been extensively studied with relation to the inhomogeneous Heisenberg ferromagnet model [17]. Note the appearance of the hidden variable μ in the equation of motion, which is the distinguishing feature of the present formalism compared to the previous studies on the inhomogeneous Heisenberg ferromagnet model. The appearance of μ in the following two systems related to the mKdV and Maxwell–Bloch equation seems to be new also. But we want to note that the soliton surface study for surfaces of non-constant curvature in [25] contains the hidden variable in its equation of motion. In the case of $\mu \to \infty$, the equation of motion in equation (34) reduces to the following form

$$\partial_t \mathbf{r} = x^2 \partial_s^2 \mathbf{r} \times \partial_s \mathbf{r} = \frac{1}{x} \partial_x^2 \mathbf{r} \times \partial_x \mathbf{r}, \tag{35}$$

which is the conventional equation of motion, obtained from equation (9) using the standard Hasimoto transformation. The generalized Hasimoto transformation and its relation with the result in equation (34) will be explained in the following section.

Example 2: the mKdV equation

In this case, $J(x) = 1$, $K(t) = 4/[-8(\mu + t)]^{3/2}$ and $\Psi = \Psi^* = \frac{1}{2}\kappa(x,t)$ (i.e. $\theta = 0$). Then equations (32) and (10) gives

$$\partial_t r = \frac{3}{2(\mu + t)} r + 12\lambda^2 x \hat{t} - 4\lambda\kappa x \hat{b} - \frac{1}{2}\partial_x \left(x \int \kappa^2 \, \mathrm{d}x \right) \hat{t} - \partial_x (x\kappa)\hat{n}. \tag{36}$$

This equation can be written in terms of the curve variable $\mathbf{r}$ as

$$\begin{aligned} \partial_t \mathbf{r} = {} & \frac{3}{2(\mu + t)} \mathbf{r} - \left(\frac{3x}{2(\mu + t)} + \frac{1}{2} \int \partial_x^2 \mathbf{r} \cdot \partial_x^2 \mathbf{r} \, \mathrm{d}x + \frac{3}{2} x \partial_x^2 \mathbf{r} \cdot \partial_x^2 \mathbf{r} \right) \partial_x \mathbf{r} \\ & - 6 \frac{x}{\sqrt{-8(\mu + t)}} \partial_x^2 \mathbf{r} \times \partial_x \mathbf{r} - \partial_x^2 \mathbf{r} - x \partial_x^3 \mathbf{r}. \end{aligned} \tag{37}$$

As in the NLSE case, this equation has the $\mu \to -\infty$ limit, which can be obtained from equation (13) using the standard Hasimoto transformation.

Example 3: the Maxwell–Bloch equation

In this case, $J(x) = 1$, $K(t) = 1/(2\sqrt{\mu + 2ct})$ and $\Psi = \frac{1}{2}\kappa(x,t)\exp(\mathrm{i}\theta)$. Then equations (32) and (14) give

$$\partial_t r = \frac{c}{\mu + 2ct} r + \frac{1}{4\lambda^2} \Phi^{-1} M \Phi. \tag{38}$$

Unlike the previous cases of NLSE and mKdV, the right-side part of equation (38) cannot be written down in terms of r. Instead, we take here

$$\partial_x \partial_t r = \partial_t \partial_x r = \frac{1}{4\mathrm{i}\lambda} \Phi^{-1} [\sigma_3, M] \Phi = \frac{1}{4\mathrm{i}\sqrt{\mu + 2ct}} [\partial_x r, 4(\mu + 2ct)\partial_t r - 4cr], \tag{39}$$

where we use equation (38) in the last step. Conventional case of constant spectral parameter can be obtained by taking $c = 0$.

2.4. Lamb formalism with the generalized Hasimoto transformation

In the previous sections, we obtain the equations of motion of space curves by using the associated linear problem of the integrable equations. In this section, we reverse the process and obtain the integrable equations by using the properties of space curves, especially the three Serret–Frenet equations. This is the Lamb formalism, which investigated the equation of motion of space curves and find their connection with the sine-Gordon and Hirota equation

[4]. This work was an extension of the result obtained by Hasimoto, and used the Hasimoto transformation which connects the curvature and torsion with the variable Ψ.

We first introduce a generalized form of the Hasimoto transformation. Equations (23) and (29) give

$$\Psi = \frac{J(x)}{2}\kappa(x,t)\exp\left[\mathrm{i}\int\{2\lambda(x,t)+J(x)\tau(x,t)\}\,\mathrm{d}x\right], \tag{40}$$

which reduces to the standard Hasimoto transformation by taking $J(x) = 1$ and $\lambda = 0$. Note that the Hasimoto transformation relates the field variable $\Psi(x,t)$ of the integrable equation to the curvature $\kappa(x,t)$ and the torsion $\tau(x,t)$ at a point x on a single curve in space at a time t. This point of view is still maintained in the present formalism. Following Lamb's procedure, we introduce

$$\mathbf{N} = (\mathbf{n}+\mathrm{i}\mathbf{b})\exp\left[\mathrm{i}\int\{2\lambda(x,t)+J(x)\tau(x,t)\}\,\mathrm{d}x\right]. \tag{41}$$

In a matrix form,

$$\begin{aligned} N &= (\hat{n}+\mathrm{i}\hat{b})\exp\left[\mathrm{i}\int\{2\lambda(x,t)+J(x)\tau(x,t)\}\,\mathrm{d}x\right] \\ &= (\hat{n}+\mathrm{i}\hat{b})\exp[\mathrm{i}\theta(x,t)] = 2\Phi^{-1}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\Phi, \end{aligned} \tag{42}$$

where we have used equations (41), (25) and (28). Then, the three Serret–Frenet equations in section 2.2 and equations (40), (41) give

$$\partial_x\mathbf{N} = J(x)\partial_s\mathbf{N} = -2\Psi\mathbf{t}+2\mathrm{i}\lambda\mathbf{N}, \qquad \partial_x\mathbf{t} = J(x)\partial_s\mathbf{t} = \Psi^*\mathbf{N}+\Psi\mathbf{N}^*. \tag{43}$$

By using the properties $\mathbf{N}\cdot\mathbf{N}^* = 2, \mathbf{N}\cdot\mathbf{t} = \mathbf{N}^*\cdot\mathbf{t} = \mathbf{N}\cdot\mathbf{N} = 0$, the temporal evolutions of $\mathbf{t}$, $\mathbf{n}$ can be expressed as following,

$$\partial_t\mathbf{N} = \mathrm{i}R\mathbf{N}+\gamma\mathbf{t}, \qquad \partial_t\mathbf{t} = -\tfrac{1}{2}(\gamma^*\mathbf{N}+\gamma\mathbf{N}^*), \tag{44}$$

where the real $R(x,t)$ and complex $\gamma(x,t)$ will be determined as following. By equating $\partial_x\partial_t\mathbf{N} = \partial_t\partial_x\mathbf{N}$ and $\partial_x\partial_t\mathbf{t} = \partial_t\partial_x\mathbf{t}$ using equations (43) and (44), one finds

$$\partial_t\Psi+\tfrac{1}{2}\partial_x\gamma-\mathrm{i}(\lambda\gamma+R\Psi) = 0, \qquad \partial_x R = \mathrm{i}(\gamma\Psi^*-\gamma^*\Psi)+2\partial_t\lambda. \tag{45}$$

By choosing explicit forms of R and γ, expressed in terms of Ψ and its spatial variables, such that they satisfy equation (45), we can obtain integrable equations having variable spectral parameter. According to the choice of R and γ, various integrable equations can be resulted. Here we show explicit forms of R and γ, which give the integrable equations of the previous sections.

Example 1: the NLSE

The NLSE corresponds to taking

$$R = 4\int\frac{|\Psi|^2}{x}\,\mathrm{d}x+2|\Psi|^2-4\lambda^2, \qquad \gamma = -2\mathrm{i}\partial_x\Psi-2\mathrm{i}\frac{\Psi}{x}+4\lambda\Psi. \tag{46}$$

Inserting these expressions into the second equation of (45) gives

$$4\lambda\partial_x\lambda+\partial_t\lambda = 0, \tag{47}$$

while the first equation gives

$$\left(\partial_x\lambda-\frac{\lambda}{x}\right)\Psi = 0, \tag{48}$$

as well as the NLSE having the variable spectral parameter, equation (9). Note that there does not appear any dependence on λ in the NLSE.

Example 2: the mKdV equation

This equation corresponds to taking

$$\begin{aligned} R &= 8\lambda^3 x - 4\lambda\partial_x\left(x\int\Psi^2\,\mathrm{d}x\right), \\ \gamma &= 2\partial_x^2(x\Psi) + 4\Psi\partial_x\left(x\int\Psi^2\,\mathrm{d}x\right) + 4\mathrm{i}\lambda\partial_x(x\Psi) - 8\lambda^2 x\Psi. \end{aligned} \tag{49}$$

Inserting these expressions into equations (45) gives

$$2\mathrm{i}\partial_x\lambda\partial_x(x\Psi) = 0, \qquad 8\lambda^3 + 24\lambda^2 x\partial_x\lambda - 4\partial_x\lambda\partial_x\left(x\int\Psi^2\,\mathrm{d}x\right) = 2\partial_t\lambda, \tag{50}$$

as well as the mKdV equation having the variable spectral parameter, equation (13). Equations (50) give the two equations for $\lambda(x,t)$ in equation (11).

Example 3: the Maxwell–Bloch equation

This equation corresponds to taking

$$R = -\frac{1}{2\lambda}\Delta, \qquad \gamma = \frac{\mathrm{i}}{2\lambda}\rho. \tag{51}$$

Inserting these expressions into equations (45) gives the equations for λ in equation (15), as well as the Maxwell–Bloch equation having the variable spectral parameter, equation (17). In fact, the second equation of (44) gives

$$\partial_t\partial_x r = \frac{\mathrm{i}}{2\lambda}\Phi^{-1}\begin{pmatrix}0 & -\rho\\ \rho^* & 0\end{pmatrix}\Phi = -\frac{\mathrm{i}}{4\lambda}\Phi^{-1}[\sigma_3, M]\Phi, \tag{52}$$

where we have used equation (42). Equation (52) is just the equation of motion for space curve in equation (39).

In the above derivation, we can see that R and γ are related to the matrix elements of V in equation (1). This fact can be easily understood when we note that equation (42) gives

$$\partial_t N = 2\Phi^{-1}\left[\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, V\right]\Phi = -2V_{1,2}\hat{t} + 2V_{1,1}N, \tag{53}$$

which gives $\gamma = -2V_{1,2}$ and $R = -2iV_{1,1}$. Similarly,

$$\partial_t\hat{t} = \Phi^{-1}[\sigma_3, V]\Phi = V_{1,2}N^* - V_{2,1}N = -\frac{\gamma}{2}N^* - \frac{\gamma^*}{2}N. \tag{54}$$

3. Curve motions

3.1. Kelvin-type curve

In this section, we will describe the Kelvin-type curve motion induced from the NLSE. The Kelvin-type motion in the curve dynamics corresponds to the solution of plane-wave type Ψ in the integrable equations. In the NLSE case,

$$\Psi = \mathrm{i}\frac{k}{x}\exp\left(\mathrm{i}\frac{x^2}{4t+\alpha}\right), \tag{55}$$

where α and k are constants. The corresponding solution in the curve dynamics is obtained as following. First we determine Φ for the nontrivial solution Ψ in equation (55). To find the solution Φ of the linear equation (1), we first define $\hat{\Phi} \equiv \Phi_0^{-1}\Phi$ where

$$\Phi_0 = \begin{pmatrix} \exp\left(\frac{\mathrm{i}}{2}\eta\right) & 0 \\ 0 & \exp\left(-\frac{\mathrm{i}}{2}\eta\right) \end{pmatrix}, \qquad \eta = \frac{x^2}{4t+\alpha}. \tag{56}$$

Then the linear equation (1) is rewritten as a linear equation for $\hat{\Phi}$,

$$\begin{aligned} &\partial_x \hat{\Phi} - \mathrm{i}\frac{k}{x}\left(\sigma_1 - \frac{x^2(4\mu-\alpha)}{4(\mu+t)(4t+\alpha)k}\sigma_3\right)\hat{\Phi} = 0 \\ &\partial_t \hat{\Phi} + \mathrm{i}\frac{k(4\mu+8t+\alpha)}{2(\mu+t)(4t+\alpha)}\left(\sigma_1 - \frac{x^2(4\mu-\alpha)}{4(\mu+t)(4t+\alpha)k}\sigma_3\right)\hat{\Phi} = 0. \end{aligned} \tag{57}$$

It is now easy to integrate equation (57), which results in

$$\hat{\Phi} = \begin{pmatrix} c_1 y\left\{I_{-\frac{1}{2}+\frac{\mathrm{i}}{2}k}(Y) + I_{\frac{1}{2}+\frac{\mathrm{i}}{2}k}(Y)\right\} + c_2 y\left\{K_{-\frac{1}{2}+\frac{\mathrm{i}}{2}k}(Y) - K_{\frac{1}{2}+\frac{\mathrm{i}}{2}k}(Y)\right\} \\ c_1 y\left\{I_{-\frac{1}{2}+\frac{\mathrm{i}}{2}k}(Y) - I_{\frac{1}{2}+\frac{\mathrm{i}}{2}k}(Y)\right\} + c_2 y\left\{(K_{-\frac{1}{2}+\frac{\mathrm{i}}{2}k}(Y) + K_{\frac{1}{2}+\frac{\mathrm{i}}{2}k}(Y)\right\} \end{pmatrix}, \tag{58}$$

where $y = x/\sqrt{(\mu+t)(4t+\alpha)}$, $Y = -\mathrm{i}(4\mu-\alpha)y^2/8$, and I_ν, K_ν are modified Bessel functions. In the following, for simplicity, we treat the case $c_1 = 1, c_2 = 0$. A 2×2 regular matrix $\hat{\Phi}_{\text{reg}}$, which is also the solution of equation (1), can be constructed from the column matrix $\hat{\Phi}$ in equation (58) as

$$\hat{\Phi}_{\text{reg}} = N_\Phi \begin{pmatrix} \hat{\Phi}_1 & -\hat{\Phi}_2^* \\ \hat{\Phi}_2 & \hat{\Phi}_1^* \end{pmatrix}, \tag{59}$$

where N_Φ is a normalization factor which makes $\mathrm{Det}(\hat{\Phi}_{\text{reg}}) = 1$. Explicitly,

$$\hat{\Phi}_{\text{reg}} = N_\Phi \begin{pmatrix} y\left\{I_{-\frac{1}{2}+\frac{\mathrm{i}}{2}k}(Y) + I_{\frac{1}{2}+\frac{\mathrm{i}}{2}k}(Y)\right\} & -y\left\{I_{-\frac{1}{2}-\frac{\mathrm{i}}{2}k}(-Y) - I_{\frac{1}{2}-\frac{\mathrm{i}}{2}k}(-Y)\right\} \\ y\left\{I_{-\frac{1}{2}+\frac{\mathrm{i}}{2}k}(Y) - I_{\frac{1}{2}+\frac{\mathrm{i}}{2}k}(Y)\right\} & y\left\{I_{-\frac{1}{2}-\frac{\mathrm{i}}{2}k}(-Y) + I_{\frac{1}{2}-\frac{\mathrm{i}}{2}k}(-Y)\right\} \end{pmatrix}. \tag{60}$$

Now, the space curve $\mathbf{r}$ is obtained using equation (19) with $\Phi = \Phi_0\hat{\Phi}_{\text{reg}}$. Explicitly, the result is

$$\begin{aligned} r_1 &= \frac{1}{D}\left\{\frac{8k(\mu+t)(4t+\alpha)}{\alpha-4\mu}(AB + A^*B^*) + x^2(A^2 + A^{*2} - B^2 - B^{*2})\right\} \\ r_2 &= \frac{-\mathrm{i}}{D}\left\{\frac{8k(\mu+t)(4t+\alpha)}{\alpha-4\mu}(AB - A^*B^*) + x^2(A^2 - A^{*2} - B^2 + B^{*2})\right\} \\ r_3 &= \frac{1}{D}\left\{\frac{8k(\mu+t)(4t+\alpha)}{\alpha-4\mu}(|B|^2 - |A|^2) + 2x^2(AB^* + A^*B)\right\}, \end{aligned} \tag{61}$$

where $A = I_{\frac{1}{2}+\frac{\mathrm{i}}{2}k}(Y)$, $B = I_{-\frac{1}{2}+\frac{\mathrm{i}}{2}k}(Y)$ and $D = 4(|A|^2 + |B|^2)$. This solution satisfies the curve equation in equation (34), which is checked explicitly using the Mathematica. Equation (61) gives the torsion and curvature $\kappa = 2k/x^2$, $\tau = (\alpha - 4\mu)/[2(\mu+t)(4t+\alpha)]$. They can also be obtained from Ψ in equation (55) by using the generalized Hasimoto transformation in equation (40). Figure 1 shows an example of Kelvin-type curve, which is obtained by using Mathematica for parameters $\mu = k = \alpha = 1$.

3.2. Solitonic curves

The one-soliton solution of integrable equations having variable spectral parameter was obtained using a generalized form of the Darboux transformation (DT) in [27]. Similarly, we can use the DT to obtain new solution $\Phi^{[N]}$ of the linear equation (1) corresponding to the

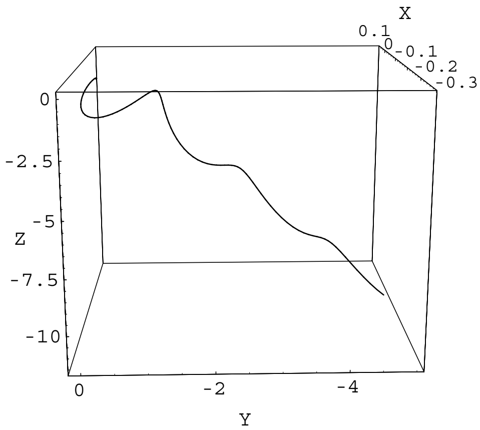

Figure 1. Curve of Kelvin-type from the NLSE having variable spectral parameter: shape at $t = 0$ as x varies from -5 to 5.

one-soliton Ψ, from the trivial Φ which is the solution corresponding to $\Psi = 0$. The DT is given by

$$\Phi^{[N]} = S[\lambda - \lambda_1^* - (\lambda_1 - \lambda_1^*)P]\Phi \equiv S[\lambda - \sigma]\Phi, \tag{62}$$

where S is introduced to make $\mathrm{Det}(\Phi^{[N]}) = 1$. Here the projection operator P ($P^2 = P$) is defined as

$$P = \frac{\Phi_1 \Phi_1^\dagger}{\Phi_1^\dagger \Phi_1}, \tag{63}$$

where the column matrix Φ_1 is a solution Φ of the linear equation in equation (1) at a specific value of the DT parameter $\lambda = \lambda_1$ [28]. By inserting these expressions into equation (1) with $U_m \to U_m^{[N]}$, $V_n \to V_n^{[N]}$, $\Phi \to \Phi^{[N]}$, we can obtain following relations,

$$\sum \lambda^m U_m^{[N]}(\lambda - \sigma) = (\lambda - \sigma) \sum \lambda^m U_m + (\partial_x S) S^{-1}(\lambda - \sigma) + \partial_x(\lambda - \sigma), \tag{64}$$

and a similar relation for V_n. Equation (64) will be used to calculate the new one-solitonic $\Psi^{[N]}$ from the trivial solution $\Psi = 0$. $\Phi^{[N]}$ will be used to obtain the one-solitonic curve using equation (19).

Here we explicitly calculate the space curves $\mathbf{r}$ and their curvatures and torsions corresponding to the one-solitonic solutions of the integrable equations considered before.

Example 1: the NLSE

The solution Φ of the Lax equation (1) for the trivial solution $\Psi = 0$ is

$$\Phi = \begin{pmatrix} \exp[\mathrm{i}x\lambda(x,t)/2] \\ -\exp[-\mathrm{i}x\lambda(x,t)/2] \end{pmatrix}. \tag{65}$$

Then the projector P in equation (63) becomes

$$P = \begin{pmatrix} 1/[1+\exp(-2l_i Y)] & -\frac{1}{2}\exp[\mathrm{i}(l_r+t)Y]\,\mathrm{sech}(4Y) \\ -\frac{1}{2}\exp[-\mathrm{i}(l_r+t)Y]\,\mathrm{sech}(4Y) & 1/[1+\exp(2l_i Y)] \end{pmatrix}, \tag{66}$$

where the DT parameter is taken to be $\lambda_1 = x/[4(l_r + \mathrm{i}l_i + t)]$, (see equation (8)) and $Y = x^2/\left[4\left(l_i^2 + l_r^2 + t^2 + 2l_r t\right)\right]$. Then

$$\Phi^{[N]} = S\begin{pmatrix}\Phi_1^{[N]}\\ \Phi_2^{[N]}\end{pmatrix},$$

$$\begin{aligned}\Phi_1^{[N]} &= \frac{x}{4}\exp\left[\mathrm{i}\frac{x^2}{8(\mu+t)}\right]\left(\frac{1}{\mu+t} - \frac{1}{l_r - \mathrm{i}l_i + t}\right) + \frac{\mathrm{i}}{x}l_i Y\left(\exp\left[\mathrm{i}\frac{x^2}{8(\mu+t)} + l_i Y\right]\right.\\ &\qquad \left. + \exp\left[-\mathrm{i}\frac{x^2}{8(\mu+t)} + \mathrm{i}(l_r+t)Y\right]\right)\mathrm{sech}(l_i Y),\\ \Phi_2^{[N]} &= -\frac{x}{4}\exp\left[-\mathrm{i}\frac{x^2}{8(\mu+t)}\right]\left(\frac{1}{\mu+t} - \frac{1}{l_r - \mathrm{i}l_i + t}\right) - \frac{\mathrm{i}}{x}l_i Y\left(\exp\left[-\mathrm{i}\frac{x^2}{8(\mu+t)} - l_i Y\right]\right.\\ &\qquad \left. + \exp\left[\mathrm{i}\frac{x^2}{8(\mu+t)} - \mathrm{i}(l_r+t)Y\right]\right)\mathrm{sech}(l_i Y),\end{aligned} \tag{67}$$

with

$$S = \sqrt{8\frac{l_i^2 + l_r^2 + t^2 + 2l_r t}{l_i^2 + l_r^2 + \mu^2 - 2l_r\mu}\frac{\mu+t}{x}}, \tag{68}$$

and λ and μ are related by equation (8). By evaluating $U_0^{[N]}$ using equations (64), (66), (62), we obtain the one-soliton solution of NLSE,

$$\Psi^{[N]} = -\frac{2}{x}l_i Y\,\mathrm{sech}(l_i Y)\exp[\mathrm{i}(l_r+t)Y]. \tag{69}$$

To obtain $\mathbf{r}$ using equation (19), we need a 2×2 regular matrix Φ_{reg}. It can be constructed from equation (67) as

$$\Phi_{\mathrm{reg}} = \begin{pmatrix}\Phi_1^{[N]} & -\Phi_2^{[N]*}\\ \Phi_2^{[N]} & \Phi_1^{[N]*}\end{pmatrix}, \tag{70}$$

which also satisfies the Lax equation (1) with $\lambda = x/[4(\mu+t)]$ for real μ. Then equation (19) gives

$$\begin{aligned} r_1 &= \frac{x^2}{2} - \frac{4l_i(\mu+t)^2}{l_i^2 + (l_r-\mu)^2}\tanh(l_i Y),\\ r_2 &= \frac{4l_i(\mu+t)^2}{l_i^2 + (l_r-\mu)^2}\mathrm{sech}(l_i Y)\sin\left[\frac{l_i^2 + (l_r-\mu)(l_r+t)}{\mu+t}Y\right],\\ r_3 &= -\frac{4l_i(\mu+t)^2}{l_i^2 + (l_r-\mu)^2}\mathrm{sech}(l_i Y)\cos\left[\frac{l_i^2 + (l_r-\mu)(l_r+t)}{\mu+t}Y\right].\end{aligned} \tag{71}$$

Note that μ is resulted from the variable spectral parameter, and it characterizes the curve equation. On the other hand, l_r, l_i are parameters which characterize the one-soliton solution. The curvature and torsion of the curve are $\kappa = l_i\,\mathrm{sech}(l_i Y)\big/\big(l_i^2 + l_r^2 + t^2 + 2l_r t\big)$ and $\tau = \big[l_i^2 + (l_r-\mu)(l_r+t)\big]\big/\big[2(\mu+t)\big(l_i^2 + l_r^2 + t^2 + 2l_r t\big)\big]$. They can also be obtained from Ψ in equation (69) by using the generalized Hasimoto transformation in equation (40). Figure 2 shows an example of one-solitonic curve, which is obtained for parameters $\mu = l_r = 1, l_i = 10$.

Example 2: the mKdV equation

A similar calculation as in the NLSE case gives following equations instead of equation (67),

$$\begin{aligned}\Phi_1^{[N]} &= \exp(\mathrm{i}\lambda x)[\lambda + \mathrm{i}\beta\tanh(2\beta x)] - \mathrm{i}\beta\exp(-\mathrm{i}\lambda x)\,\mathrm{sech}(2\beta x),\\ \Phi_2^{[N]} &= \exp(-\mathrm{i}\lambda x)[-\lambda + \mathrm{i}\beta\tanh(2\beta x)] + \mathrm{i}\beta\exp(\mathrm{i}\lambda x)\,\mathrm{sech}(2\beta x),\end{aligned} \tag{72}$$

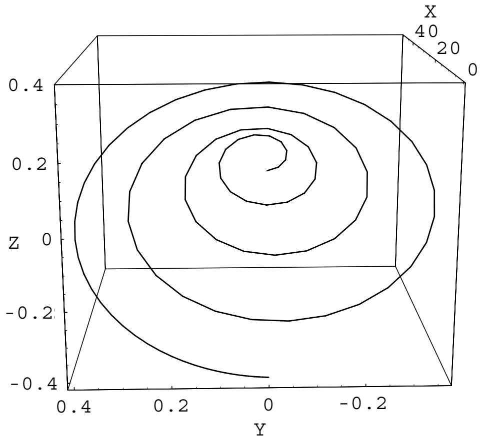

Figure 2. Curve of one-soliton induced from the NLSE having variable spectral parameter: shape at $t = 0$ as x varies from -10 to 10.

with

$$S = \frac{1}{\sqrt{2(\lambda^2 + \beta^2)}}, \tag{73}$$

where the DT parameter is $\lambda_1 \equiv \mathrm{i}\beta = \mathrm{i}\beta(t) = 1/\sqrt{-8(\mu_1 + t)}$ for real μ_1 satisfying $-\mu_1 < t$. Here, $\lambda = \lambda(t) = 1/\sqrt{-8(\mu + t)}$ in equation (12). This form of λ_1 is required to have a real one-soliton solution, $\Psi^{[N]} = \Psi^{[N]*}$. In this case, the one-soliton solution is

$$\Psi^{[N]} = 2\beta(t)\,\mathrm{sech}[2\beta(t)x]. \tag{74}$$

Similar procedure as in the NLSE case gives

$$\begin{aligned} r_1 &= x - \frac{\beta(t)\tanh[2\beta(t)x]}{\lambda(t)^2 + \beta(t)^2}, \\ r_2 &= -\frac{\beta(t)}{\lambda(t)^2 + \beta(t)^2}\,\mathrm{sech}[2\beta(t)x]\sin[2\lambda(t)x], \\ r_3 &= \frac{\beta(t)}{\lambda(t)^2 + \beta(t)^2}\,\mathrm{sech}[2\beta(t)x]\cos[2\lambda(t)x]. \end{aligned} \tag{75}$$

The curvature of the curve is $\kappa = 4\beta(t)\,\mathrm{sech}[2\beta(t)x]$, while the torsion is $\tau = 2\lambda(t)$.

Example 3: the Maxwell–Bloch equation

The solution Φ of the Lax equation (1) for the trivial solution $\Psi = \rho = 0$, $\Delta = -4cx$ is again given by equation (65) without the factor 2, where $\lambda(t)$ is given by equation (16). Then, similar calculation as in the case of mKdV equation gives

$$\begin{aligned} \Phi_1^{[N]} &= \exp(\mathrm{i}\lambda x)[\lambda - \alpha + \mathrm{i}\beta\tanh(2\beta x)] - \mathrm{i}\beta\exp(2\mathrm{i}\alpha x - \mathrm{i}\lambda x)\,\mathrm{sech}(2\beta x), \\ \Phi_2^{[N]} &= \exp(-\mathrm{i}\lambda x)[-\lambda + \alpha + \mathrm{i}\beta\tanh(2\beta x)] + \mathrm{i}\beta\exp(\mathrm{i}\lambda x - 2\mathrm{i}\alpha x)\,\mathrm{sech}(2\beta x), \end{aligned} \tag{76}$$

with

$$S = \frac{1}{\sqrt{2(\lambda^2 + \beta^2 + \alpha^2 - 2\lambda\alpha)}}, \tag{77}$$

where the DT parameter is $\lambda_1 \equiv \alpha + \mathrm{i}\beta = \alpha(t) + \mathrm{i}\beta(t) = \sqrt{\mu_1 + 2ct}$ for a complex μ_1, while $\lambda = \lambda(t) = \sqrt{\mu + 2ct}$ for a real μ. The one-soliton solution is

$$\Psi^{[N]} = 4\beta(t)\,\mathrm{sech}[2\beta(t)x]\exp[2\mathrm{i}\alpha(t)x]. \tag{78}$$

Especially, equation (64) for V at λ^{-1} level gives

$$V_{-1}^{[N]} = \sigma V_{-1}\sigma^{-1} - c\sigma^{-1} + S^{-1}\partial_t S\lambda, \tag{79}$$

which gives the new $\Delta^{[N]}$ and $\rho^{[N]}$,

$$\begin{aligned}\Delta^{[N]} &= \frac{4c}{\alpha^2+\beta^2}[2\beta^2 x\,\mathrm{sech}^2(2\beta x) + \beta\tanh(2\beta x)] - 4cx,\\ \rho^{[N]} &= -\frac{4\beta c}{\alpha^2+\beta^2}[2\mathrm{i}\alpha x - 1 + 2\beta x\tanh(2\beta x)]\exp(2\mathrm{i}\alpha x)\,\mathrm{sech}(2\beta x),\end{aligned} \tag{80}$$

where we have used equations (14), (62), (63), 76) and (77).

Similar procedure as in the case of mKdV equation gives $\mathbf{r}$ as in equation (75) with the replacement $\lambda(t) \to \lambda(t) - \alpha(t)$. The curvature of the curve is $\kappa = 4\beta(t)\,\mathrm{sech}[2\beta(t)x]$, while the torsion is $\tau = 2[\alpha(t) - \lambda(t)]$,

4. Discussion

In this paper, we have developed the 'space curve' formulation induced from integrable equations having variable spectral parameters. The integrable equations and curve motions are related through a generalized form of the Hasimoto transformation. This transformation provides an explicit method to calculate curve motions moving in an inhomogeneous space. The form of inhomogeneity is determined by the spacetime dependence of the variable spectral parameter.

The present formalism can be easily adapted to describe the duality between the spin motions and the integrable equations having variable spectral parameters. In fact, the curve $\mathbf{r}$ is related to the spin $\mathbf{S}$ via $\mathbf{S} = \partial_x\mathbf{r}$, which is just the unit tangent vector. Thus, by using the present formalism, we can easily find the equations of the site-dependent Heisenberg spin chain described by the inhomogeneous NLSE, which should generalize the equations of the site-dependent Heisenberg chain in the literature by including the inhomogeneity related to the variable spectral parameter [23].

Finally we note that a slight generalization of the present formulation could describe the curve motions related to the Darboux covariant integrable system which has U_2 and/or U_3 terms in the Lax pair U in equation (1) [29]. In this case, the Hasimoto transformation should be generalized to a more complex form. The equation for the curve motion involves inhomogeneous terms which, themselves, satisfy separate nonlinear equations. These types of curve motions should be interesting as they include the solitonic configurations that can be easily observed in nature. This will be reported in a separate manuscript.

Acknowledgments

This work was supported by Korea Research Foundation Grant (KRF-2003-070-C00011).

References

[1] See for instance, Brower R C, Kessler D A, Koplik J and Levine H 1984 *Phys. Rev.* A **29** 1335
Sethian J A 1991 *J. Diff. Geom.* **31** 131
[2] Hasimoto H 1972 *J. Fluid Mech.* **51** 477
[3] Lakshmanan M, Ruijgrok T W and Thompson C J 1976 *Physica* A **84** 577
Lakshmanan M 1977 *Phys. Lett.* A **61** 53
[4] Lamb G L 1977 *J. Math. Phys.* **18** 1654
[5] Goldstein R E and Petrich D M 1991 *Phys. Rev. Lett.* **67** 3203

Goldstein R E and Petrich D M 1992 *Phys. Rev. Lett.* **69** 555
[6] Nakayama K, Segur H and Wadati M 1992 *Phys. Rev. Lett.* **69** 2603
[7] Sym A 1983 *Lett. Nuovo Cimento* **36** 307
Sym A 1985 Soliton surfaces and their applications *Geometric Aspects of the Einstein Equations and Integrable Systems* ed R Martini (*Lecture Notes in Physics* vol 239) (Berlin: Springer) pp 154–231
[8] Cieśliński J 1997 *J. Math. Phys.* **38** 4255 and references therein
[9] Shin H J 2001 *J. Phys. A: Math. Gen.* **34** 3169
[10] See for instance, Bobenko A I 1993 *Harmonic Maps and Integrable Systems* ed A P Fordy and J C Wood (Braunschweig: Vieweg)
Balakrishnan R 1997 *Pramana J. Phys.* **48** 189
[11] Park Q H and Shin H J 1999 *Phys. Lett.* B **454** 259
Lee K M, Park Q H and Shin H J 1999 *Nucl. Phys.* B **563** 461
[12] Burtsev S P, Mikhailov A V and Zakharov V E 1987 *Theor. Math. Phys.* **70** 227
[13] Calogero F and Degasperis A 1978 *Commun. Math. Phys.* **63** 155
[14] Rice M J, Bishop A R, Krumhausel J A and Tullinger S E Z 1976 *Phys. Rev. Lett.* **36** 432
Kaup D J and Newell A C 1978 *Phys. Rev* B **18** 5162
[15] Clarkson P A and Cosgrove C M 1987 *J. Phys. A: Math. Gen.* **20** 2003
[16] Balakrishnan R 1985 *Phys. Rev.* A **32** 1114
[17] Balakrishnan R and Guha P 1996 *J. Math. Phys.* **37** 3651
Balakrishnan R 1982 *Phys. Lett* **92** 243
[18] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (*London Mathematical Society Lecture Note Series* vol 149) (Cambridge: Cambridge University Press)
[19] Vinoj M N, Kuriakose V C and Porsezian K 2001 *Chaos, Solitons Fractals* **12** 2569
[20] Porsezian K 2000 *J. Mod. Opt.* **47** 1635
[21] Shin H J 2002 *Phys. Lett.* **294** 199
[22] Shin H J 2003 *J. Phys. Soc. Japan* **72** 496
[23] Balakrishnan R 1982 *J. Phys. C: Solid State Phys.* **15** L1305
Lakshmanan M and Bullough R K 1980 *Phys. Lett.* A **80** 287
Cieśliński J, Goldstein P and Sym A 1994 *J. Phys. A: Math. Gen.* **27** 1645
[24] Porsezian K 1997 *Phys. Rev.* E **55** 3785
[25] Levi D and Sym A 1990 *Phys. Lett.* A **149** 381
[26] Burtsev S P and Gabitov I R 1994 *Phys. Rev.* A **49** 2065
[27] Rybin A 1991 *J. Phys. A: Math. Gen.* **24** 5235
[28] Park Q H and Shin H J 2001 *Physica* D **157** 1
[29] Hayashi H and Nozaki K 1994 *J. Phys. Soc. Japan* **63** 27
Imai K 1998 *J. Phys. Soc. Japan* **67** 1811